Coherent sheaves, superconnection Riemann-Roch-Grothendieck

joint work J.-M. Bismut & Z. Wei, arXiv:2102.08129,
to appear in Progress in Mathematics 347

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Paris, Sep 6, 2023

Main results

- X: closed complex manifold.
- $K(X)$: K-group of coherent sheaves.
  - holomorphic analogue of the topological K-theory.
- $H_{\mathrm{BC}}(X,\mathbf{R})$: Bott-Chern cohomology.
  - holomorphic analogue of the de Rham cohomology.

Theorem (Bismut-S.-Wei, 2021)

1. There is a Chern character $\mathrm{ch}_{\mathrm{BC}} : K(X) \to H_{\mathrm{BC}}(X,\mathbf{R})$.
2. $\mathrm{ch}_{\mathrm{BC}}$ satisfies RRG formula for arbitrary holomorphic map $f : X \to Y$, i.e., the diagram commutes

$$\begin{array}{ccc} K(X) & \xrightarrow{f_!} & K(Y) \\ \Big\downarrow {\scriptstyle \mathrm{Td}_{\mathrm{BC}}(TX)\mathrm{ch}_{\mathrm{BC}}} & & \Big\downarrow {\scriptstyle \mathrm{Td}_{\mathrm{BC}}(TY)\mathrm{ch}_{\mathrm{BC}}} \\ H_{\mathrm{BC}}(X,\mathbf{R}) & \xrightarrow{f_*} & H_{\mathrm{BC}}(Y,\mathbf{R}) \end{array}$$

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de Rham cohomology

- $\Omega^k(X, \mathbf{R})$: smooth k-forms.
- $d : \Omega^\bullet(X, \mathbf{R}) \to \Omega^{\bullet+1}(X, \mathbf{R})$: de Rham operator.
- $d^2 = 0$.

Definition (de Rham)

$H^p_{dR}(X, \mathbf{R}) = \ker d \cap \Omega^p(X, \mathbf{R}) / d\Omega^{p-1}(X, \mathbf{R})$.

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Bott-Chern cohomology

- $\Omega^{p,q}(X,\mathbf{C})$: smooth (p,q)-forms.
- $d = \partial + \overline{\partial}$.
- Classical relation $\partial^2 = 0, \overline{\partial}^2 = 0, [\partial, \overline{\partial}] = 0$.

Definition (Bott-Chern)

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Bott-Chern vs de Rham

- Canonical morphism : $H^{p,q}_{\mathrm{BC}}(X,\mathbf{C}) \to H^{p+q}_{\mathrm{dR}}(X,\mathbf{C})$.
- If X is Kähler, $\oplus_{p+q=k} H^{p,q}_{\mathrm{BC}}(X,\mathbf{C}) \simeq H^{k}_{\mathrm{dR}}(X,\mathbf{C})$.
- In general, $H_{\mathrm{BC}}(X,\mathbf{C}) \not\simeq H_{\mathrm{dR}}(X,\mathbf{C})$ (e.g. Iwasawa manifold, Hopf manifold).

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Holomorphic vector bundles

- D: holomorphic vector bundle.
- $\nabla^{D''} : \Omega^{0,\bullet}(X, D) \to \Omega^{0,\bullet+1}(X, D)$ holomorphic structure.
  1. Leibniz rule: $\nabla^{D''}(\alpha s) = \bar{\partial}\alpha \cdot s + (-1)^{\deg \alpha} \alpha \wedge \nabla^{D''} s$
  2. $(\nabla^{D''})^2 = 0$.

Theorem (Koszul-Malgrange, Newlander-Nirenberg)

A smooth vector bundle D is holomorphic iff there is $\nabla^{D''} : \Omega^{0,\bullet}(X, D) \to \Omega^{0,\bullet+1}(X, D)$ with Leibniz rule and $(\nabla^{D''})^2 = 0$.

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Chern connection

- $(D, \nabla^{D''})$: holomorphic vector bundle.
- h^D : Hermitian metric on D.
- $\nabla^D = \nabla^{D''} + \nabla^{D'}$: Chern connection.

Chern connection

- $(D, \nabla^{D\prime\prime})$: holomorphic vector bundle.
- h^D : Hermitian metric on D.
- $\nabla^D = \nabla^{D\prime\prime} + \nabla^{D\prime}$: Chern connection. (Unique unitary connection whose antiholomorphic part is given by the holomorphic structure.)
- We have

$$(\nabla^{D\prime\prime})^2 = 0, \qquad (\nabla^{D\prime})^2 = 0, \qquad (\nabla^D)^2 = [\nabla^{D\prime\prime}, \nabla^{D\prime}]$$

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Chern-Weil & Bott-Chern theory

Definition

$$\mathrm{ch}(D, \nabla^{D''}, h^D) = \mathrm{Tr}\left[\exp(-R^D/2i\pi)\right] \in \Omega(X, \mathbf{C}).$$

Theorem (Chern-Weil, Bott-Chern)

1. $\mathrm{ch}(D, \nabla^{D''}, h^D) \in \oplus_p \Omega^{p,p}(X, \mathbf{R})$ *and d-closed.*
2. $\mathrm{ch}_{\mathrm{BC}}(D, \nabla^{D''}) = [\mathrm{ch}(D, \nabla^{D''}, h^D)] \in \underbrace{\oplus_p H^{p,p}_{\mathrm{BC}}(X, \mathbf{R})}_{H_{\mathrm{BC}}(X,\mathbf{R})}$ *is independent of* h^D*: holomorphic invariant!!*

Remark

$\mathrm{ch}_{\mathrm{dR}}(D) = [\mathrm{ch}(D, \nabla^{D''}, h^D)]_{\mathrm{dR}} \in H^{\mathrm{even}}_{\mathrm{dR}}(X, \mathbf{R})$ is a topological invariant.

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$\mathrm{ch}_{\mathrm{dR}}(D) = [\mathrm{ch}(D, \nabla^{D''}, h^D)]_{\mathrm{dR}} \in H^{\mathrm{even}}_{\mathrm{dR}}(X, \mathbf{R})$ is a topological invariant.

Chern-Weil & Bott-Chern theory

Definition

$\mathrm{ch}(D, \nabla^{D''}, h^D) = \mathrm{Tr}\left[\exp(-R^D/2i\pi)\right] \in \Omega(X, \mathbf{C})$.

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$c_{1,\mathrm{BC}}$ vs $c_{1,\mathrm{dR}}$

Example

1. *Hopf surface:* $X = \mathbf{C}^2 \backslash \{0\}/\mathbf{Z}$ *where* $n \cdot (z_1, z_2) = 2^n (z_1, z_2)$.
2. $X \simeq_{\mathrm{diffeo}} S^3 \times S^1$, $H^2_{\mathrm{dR}}(X, \mathbf{R}) = 0$ *and* $c_{1,\mathrm{dR}}(TX) = 0$.
3. $\dim H^{1,1}_{\mathrm{BC}}(X, \mathbf{R}) = 1$ *and* $c_{1,\mathrm{BC}}(TX) \neq 0$.

Complex of holomorphic vector bundles

- Complex of holomorphic vector bundles

$$0 \longrightarrow D^r \xrightarrow{v} D^{r+1} \xrightarrow{v} \dots \xrightarrow{v} D^{r'} \longrightarrow 0 \, .$$

  - D^i has a holomorphic structure $\nabla^{D^i\prime\prime}$.
  - v is holomorphic, i.e., $\left[v, \nabla^{D\prime\prime}\right] = 0$.
- $\Omega^{0,p}(X, D^q)$ has total degree $p + q$.
- $A'' = v + \nabla^{D\prime\prime} : \Omega^{0,\bullet}(X, D^\bullet) \to [\Omega^{0,\bullet}(X, D^\bullet)]^{+1}$ has total degree 1 and $(A'')^2 = 0$.
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K-theory of holomorphic vector bundles

Definition

$K^\bullet(X)$: Abelian group

- *Generators : holomorphic vector bundles.*
- *Relations: if we have a short exact sequence*

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Coherent sheaves

- The category of holomorphic vector bundles is not good.
- If $f : D \to \underline{D}$ is a holomorphic bundle map, then $\ker f$ and $\operatorname{im} f$ are not holomorphic vector bundles.
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An $\mathcal{O}_X$-complex $(\mathscr{F}^\bullet, \partial)$ has coherent cohomologies iff for any small open set $U \subset X$, there exist a complex of holomorphic vector bundles (E_U, v_U) on U, and a quasi-isomorphism

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sheaves of holo. sections in E_U

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- If X is projective, (E_U, v_U) exists globally, i.e., $U = X$.
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2. Is $\mathrm{ch}_{\mathrm{BC}}$ compatible with the direct image associated to $f : X \to Y$ (RRG)?

Bismut-S.-Wei 2021: yes !

Remark

- Grothendieck [illegible] : X, Y are projective and ch with taking in Chow groups.
- Atiyah-Hirzebruch [illegible] : ch taking values in [illegible] and RRG for immersion.
- Green, O'Brian, Toledo, Tong, Levy : [illegible].
- Grivaux (2010) : ch taking values in [illegible] and RRG for projective morphism.
- Wu (2020) : ch taking values in [illegible] and RRG for projective morphism.
- Grivaux's unicity theorem: all the constructions of Chern Character are compatible.

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Antiholomorphic superconnections

- $D^\bullet = \oplus_{i=r}^{r'} D^i$: $\mathbf{Z}$-graded smooth vector bundles on X.

Definition (Quillen 85, Block 2010)

$A'' : \Omega^{0,\bullet}(X, D^\bullet) \to [\Omega^{0,\bullet}(X, D^\bullet)]^{+1}$ of total degree 1 is called an antiholomorphic superconnection, if

1. $A'' = v_0 + \nabla^{D''} + v_2 + \ldots$ where $v_i \in \Omega^{0,i}(X, \mathrm{End}^{1-i}(D))$ and $\nabla^{D''}$ antiholo. part of some connection,
2. $(A'')^2 = 0$.

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An example

If $v_2 = v_3 = \ldots = 0$, then

$$(A'')^2 = 0 \iff v_0^2 = 0, \left[\nabla^{D''}, v_0\right] = 0, (\nabla^{D''})^2 = 0.$$

By Koszul-Malgrange/Newlander-Nirenberg, (D, v_0) is a complex of holomorphic vector bundles.

Block's Theorem

- Given $(D^\bullet, A'')$, we can define a $\mathcal{O}_X$-complex $\mathscr{E}^\bullet(D^\bullet, A'')$ by

$$U \subset X \text{ open } \to \left(\Omega^{0,\bullet}(U, D^\bullet|_U), A''|_U\right).$$

Theorem (Block 2010, Bismut-S.-Wei 2021)

1. $\mathscr{E}^\bullet(D^\bullet, A'')$ has coherent cohomologies.
2. Every $(\mathscr{F}^\bullet, \partial)$ can be "obtained" in this way.

Proof.

- Locally, after conjugason: $A'' \simeq v + \nabla''$ (extension of Koszul-Malgrange/Newlander-Nirenberg).
- $(D^\bullet, A'') \to \mathscr{E}^\bullet(D^\bullet, A'')$ defines an equivalence of categories.

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Another proof given by Bondal-Rosly 2022.

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Chern-Weil theory for (D, A'')

- h: $\mathbf{Z}$-graded Hermitian metric on $D^\bullet$.
- $A = A'' + A'$: unitary superconnection.
- $(A'')^2 = 0, (A')^2 = 0$ and $A^2 = [A'', A']$.

Definition

$\mathrm{ch}(D, A'', h) = \frac{1}{(2i\pi)^{N/2}} \mathrm{Tr_s}[\exp(-A^2)]$.

Theorem (Bismut-S.-Wei 2021)

1. $\mathrm{ch}(D, A'', h) \in \oplus_p \Omega^{p,p}(X, \mathbb{R})$ and d-closed.
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Proof of RRG: strategy

- Write $f = \pi \circ i$ where
  - $i : X \to X \times Y$ (graph of f) immersion.
  - $\pi : X \times Y \to Y$ projection.
- Thanks to $f_! = \pi_! i_!$ and $f_* = \pi_* i_*$, we need only to show the following two diagrams commute.

$$\begin{array}{ccccc}
K(X) & \xrightarrow{\quad i_! \quad} & K(X \times Y) & \xrightarrow{\quad \pi_! \quad} & K(Y) \\
\Big\downarrow {\scriptstyle \mathrm{Td}_{\mathrm{BC}}(TX)\mathrm{ch}_{\mathrm{BC}}} & & \Big\downarrow {\scriptstyle \mathrm{Td}_{\mathrm{BC}}(T(X \times Y))\mathrm{ch}_{\mathrm{BC}}} & & \Big\downarrow {\scriptstyle \mathrm{Td}_{\mathrm{BC}}(TY)\mathrm{ch}_{\mathrm{BC}}} \\
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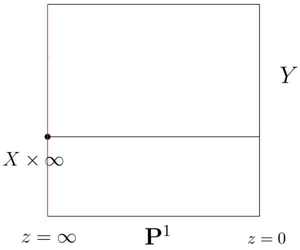
RRG for immersions : deformation to normal cone

$$W = \mathrm{Bl}_{X\times\infty}(Y \times \mathbf{P}^1).$$



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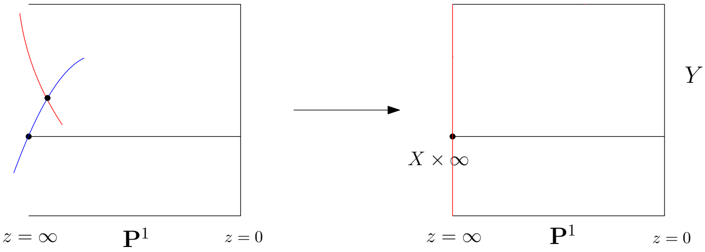

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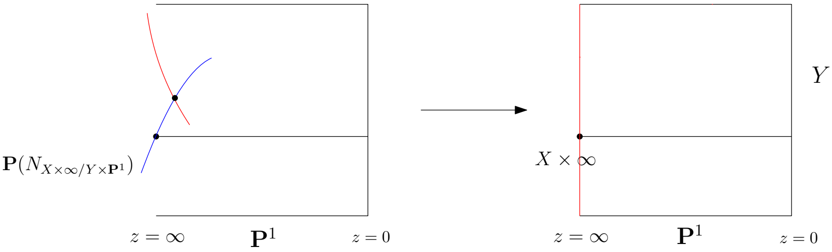

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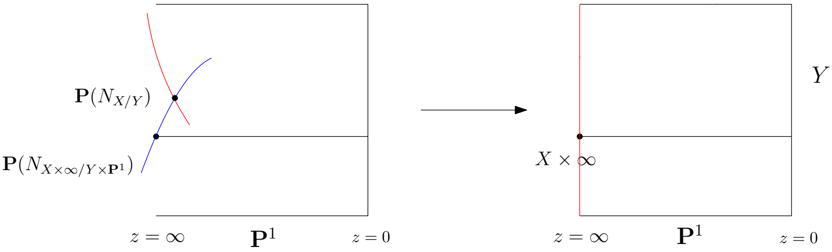

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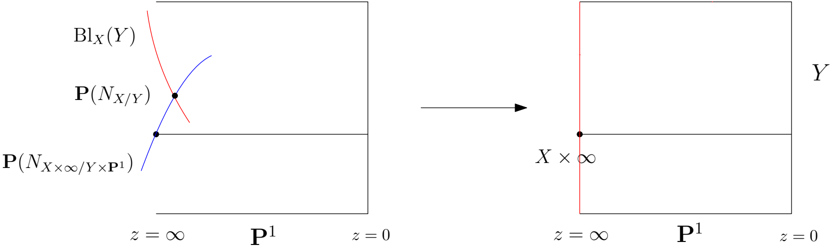

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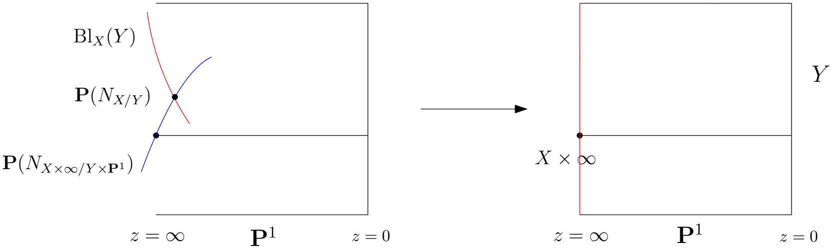

Deform an immersion $X \to Y$ to an other immersion

$$X \to \mathbf{P}(N_{X\times\infty/Y\times\mathbf{P}^1}).$$

Direct image for projection

- $\pi : M = X \times S \to S$.
- Assume $\mathscr{F} = \mathscr{E}^\bullet(D^\bullet, A'') \in K(M)$.
- We need to show

$$\mathrm{ch}_{\mathrm{BC}}(\pi_! \mathscr{F}) = \int_X \mathrm{Td}_{\mathrm{BC}}(TX) \mathrm{ch}_{\mathrm{BC}}(D^\bullet, A'') \text{ in } H_{\mathrm{BC}}(S, \mathbf{R})$$

- $\mathcal{D}^\bullet = \Omega^{0,\bullet}(X, D^\bullet|_X)$: infinite dimensional $\mathbf{Z}$-graded vector bundle on S.
- $\Omega^{0,\bullet}(S, \mathcal{D}^\bullet) = \Omega^{0,\bullet}(M, D^\bullet)$.
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- Given metrics g^D and g^{TX}, we can define an L^2-metric on $\mathcal{D} = \Omega^{0,\bullet}(X, D^\bullet|_X)$.
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spectral truncation + fibrewise Hodge theory.

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- J^{TX} complex structure on X, $\omega^X = g^{TX}(\cdot, J^{TX}\cdot)$.
- If $\overline{\partial}^X \partial^X \omega^X = 0$, by local family index theorem, as $t \to 0$,

$$\mathrm{ch}(\mathcal{D}, \mathcal{A}'', g^D, g^{TX}/t) \to \text{some limit in } \Omega(S, \mathbf{R})$$

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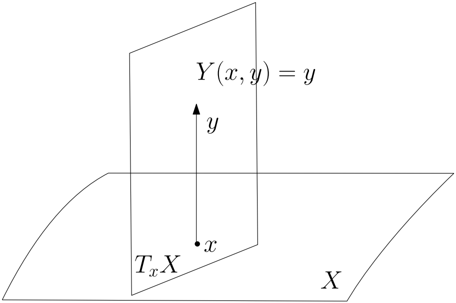
Dolbeault-Koszul resolution

- $\mathcal{X} = TX$. $Y \in C^\infty(\mathcal{X}, \pi^* TX)$.



- $i : X \to \mathcal{X}$ by zero section.
- Dolbeault-Koszul: $i_! \mathcal{O}_X = \mathscr{E}^\bullet \left(\pi^* \Lambda^\bullet (T^* X), \overline{\partial}^{\mathcal{X}} + i_Y \right)$.

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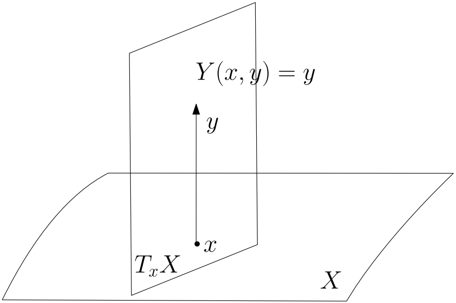

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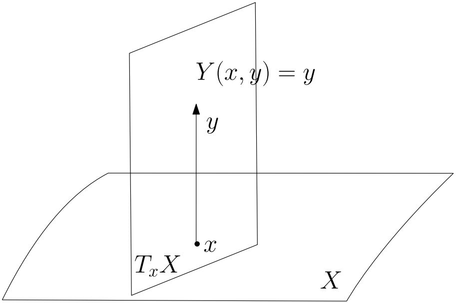

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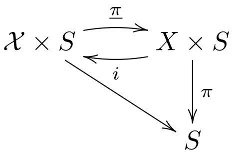
Enlarge the fibration



- $i_! \mathscr{F}$ and $\mathscr{F}$ are expected to have the same direct image on S.
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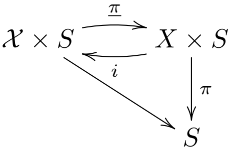

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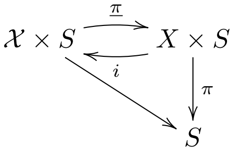

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Hypoelliptic deformation

- Infinite dimensional object on S:

$$\left(\underbrace{\Omega^{0,\bullet}\left(\mathcal{X}, \underline{\pi}^* \left(\Lambda(T^*X)\widehat{\otimes} D\right)\right)}_{\mathcal{D}}, \mathcal{A}''_Y\right)$$

- $\mathcal{D} = \Omega^{\bullet,\bullet}(X, \Omega^{0,\bullet}(TX) \otimes D)$.
- $g^D, g^{TX} \rightsquigarrow L^2$-metric on $\Omega^{0,\bullet}(TX) \otimes D \rightsquigarrow$ non degenerate Hermitian form by twisting $r : (x, Y) \to (x, -Y)$.
- $\omega^X \rightsquigarrow$ non degenerate Hermitian form $\left(\frac{i}{2\pi}\right)^{\dim X} \int_X \widetilde{\alpha} \wedge \overline{e^{-i\omega^X}\beta}$
- We get non degenerate Hermitian form on $\mathcal{D}$.
- $\mathcal{A}_Y = \mathcal{A}''_Y + \mathcal{A}'_Y$, $\mathcal{A}^2_Y$ is hypoelliptic,

$$\mathcal{A}^2_Y = \frac{1}{2}(-\Delta^V + |Y|^2 + \ldots) + \nabla_{Y^H} + i\partial\overline{\partial}\omega^X + \ldots$$

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Hypoelliptic Chern-Weil theory

- We can define $\mathrm{ch}(\mathcal{A}''_Y, g^D, g^{TX}, \omega^X)$ as before.

Theorem

1. $\mathrm{ch}(\mathcal{A}''_Y, g^D, g^{TX}, \omega^X) \in \oplus_p \Omega^{p,p}(S, \mathbf{R})$ *and* d*-closed*
2. $[\mathrm{ch}(\mathcal{A}''_Y, g^D, g^{TX}, \omega^X)] \in H_{\mathrm{BC}}(S, \mathbf{R})$ *is independent of* g^D, g^{TX}, ω^X.
3. $[\mathrm{ch}(\mathcal{A}''_Y, g^D, g^{TX}, \omega^X)] = \mathrm{ch}_{\mathrm{BC}}(\pi_! \mathscr{F}) \in H_{\mathrm{BC}}(S, \mathbf{R})$.

Proof.

Part 3 is based on the fact that the hypoelliptic curvature $\mathcal{A}_Y^2$ can be deformed to the elliptic curvature $\mathcal{A}^2$. As $b \to 0$, we have (Bismut-Lebeau 08)

$$\mathrm{ch}(\mathcal{A}''_Y, g^D, b^4 g^{TX}, \omega^X) \to \mathrm{ch}(\mathcal{A}'', g^D, g^{TX}) \text{ in } \Omega(S, \mathbf{R}).$$

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Hypoelliptic local index theorem

- If $\overline{\partial}^X \partial^X \omega^X = 0$, as $t \to 0$,

$$\text{ch}\left(\mathcal{A}''_Y, g^D, g^{TX}/t^3, \omega^X/t\right) \to \int_X \text{Td}(TX, g^{TX})\text{ch}(D, A'', g^D). \tag{3.1}$$

- If we replace ω^X by $|Y|^2\omega^X$ in the construction, as $t \to 0$,

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  without any assumption!

- The associated hypoelliptic Laplacians are
  - case (3.1): $-\frac{1}{2}\Delta^V + |tY|^2 + \ldots$ [illegible]
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Thank you for your attention !